

Constraining $f(R)$ Theories with Temporal Variation of Fine Structure Constant

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Abstract

It is well-known that $f(R)$ theories in Einstein frame is conformally equivalent to quintessence models in which the scalar field minimally couples with gravity. If there exists a matter system in Jordan frame, then it interacts with the scalar field in Einstein frame due to the conformal transformations. This interaction, in general, may lead to changes of fundamental constants. Here we will consider possible time variation of fine structure constant in a general $f(R)$ theory. We will use observational bounds on these variations and argue that it provides a criterion for constraining $f(R)$ models.

1 Introduction

Recent observations on expansion history of the universe indicate that the universe is experiencing a phase of accelerated expansion [1]. This can be interpreted as evidence either for existence of some exotic matter components or for modification of the gravitational theory. In the first route of interpretation one can take a perfect fluid with a sufficiently negative pressure, dubbed dark energy [2], to produce the observed acceleration. In the second route, however, one attributes the accelerating expansion to a modification of general relativity. A particular class of models that has recently drawn a significant amount of attention is the so-called $f(R)$ gravity models [3][4]. These models propose a modification of Einstein-Hilbert action so that the scalar curvature is replaced by some arbitrary function $f(R)$.

It is well known that conformal transformations can be used to recast $f(R)$ theories of gravity into the form of Einstein gravity together with a minimally coupled scalar field. The original variable is called Jordan conformal frame while the transformed set, whose dynamics is described by Einstein equations, is called Einstein conformal frame. If one formulates a $f(R)$ theory with some matter systems in the Jordan frame, then the conformal transformation induces a coupling of the scalar field with the matter sector in the Einstein frame. One of the important implications of such a coupling is the possible changes of constants of nature. If the effective mass of the scalar field is sufficiently small the interaction of the field with ordinary matter results in the time variation of constants of nature over cosmological time scales [5].

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Among all possibilities, we restrict ourselves to time variation of fine structure constant as a consequence of interaction of the scalar field with electrodynamics. We intend here to use the current bounds on a running fine structure constant to constrain a general $f(R)$ theory in the Einstein frame.

2 Quintessence and time variation of fine structure constant

Many years after Dirac's proposal [6] that fundamental constants may vary with time and/or space, there is now some theoretical frameworks which predict such changes. For instance, one of the most interesting low energy features of string theory is the possible presence of a massless scalar field, dilaton or moduli fields, whose vacuum expectation values define the size of the effective coupling constants. Independent of this framework, Bekenstein [7] has also formulated a theory for varying fine structure constant which has been recently generalized and applied to a cosmological setting [8][9]. Moreover, there is a large class of dark energy and quintessence models which invoke scalar fields with wavelengths comparable to the size of the universe. This scalar field may then interact with matter and radiation so that changes of its background value induce variation of coupling constants [5].

In recent years, there is an increasing interest to models concerning time variation of fine structure constant due to a non-vanishing coupling of quintessence to the electromagnetic field. In these models, one can combine cosmological data and terrestrial observations to place empirical constraints on the size of the coupling or even explore the conditions that the scalar field exhibit a tracking behavior [10][11][12]. The general form of the action in these models is the following [†]

$$S = \frac{1}{2} \int d^4x \sqrt{-g} R + \int d^4x \sqrt{-g} (L_\phi + L_m + L_{\phi F}) \quad (1)$$

where R is the curvature scalar and L_m is Lagrangian density of cosmic matter which we take it as a perfect fluid with energy density ρ_m and equation of state parameter ω , and

$$L_\phi = \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - V(\phi) \quad (2)$$

$$L_{\phi F} = -\frac{1}{4} A(\phi) F_{\mu\nu} F^{\mu\nu} \quad (3)$$

$A(\phi)$ is a dimensionless function[‡] of ϕ and $F_{\mu\nu}$ is the components of electromagnetic field tensor. For spatially flat Friedmann-Robertson-Walker (FRW) spacetime, the governing field equations for the scale factor $a(t)$ and the scalar field are

$$H^2 = \frac{1}{3} (\rho_m + \frac{1}{2} \dot{\phi}^2 + V(\phi)) \quad (4)$$

[†]We work in the unit system in which $\hbar = c = 8\pi G = 1$ and our sign convention is $(-+++)$.

[‡]We may call it a gauge function since evolution of fine structure constant strongly depends on the choice of this function.

$$\dot{H} = -\frac{1}{2}(\omega\rho_m + \dot{\phi}^2) \quad (5)$$

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV(\phi)}{d\phi} = 0 \quad (6)$$

where $H \equiv \frac{\dot{a}}{a}$. These field equations depend on the choice of the functions $V(\phi)$ and $A(\phi)$. These functions characterize the evolution of the scalar field and the effective fine structure constant which is now given by

$$\alpha(\phi) = \frac{\alpha_0}{A(\phi)} \quad (7)$$

with α_0 being its present value and $A(\phi_0) = 1$. The choice of the gauge function $A(\phi)$ is arbitrary although the choice of linear [11] and exponential [8][12] functions are usual in the literature. It should give an appropriate temporal variations of fine structure constant which is consistent with observational constraints. We bring in the following two of these constraints which we will use in the next section[§]:

1) Estimates of the age of iron meteorites ($z = 0.45$) combined with a measurement of the ratio Re/Os resulting from the decay rate $R^{187} \rightarrow Os^{187}$ gives [13]

$$\left| \frac{\Delta\alpha}{\alpha} \right| < 10^{-6} \quad (8)$$

2) We have also [14]

$$\left| \frac{\dot{\alpha}}{\alpha} \right| < 4.2 \times 10^{-15} \text{ yr}^{-1} \quad (9)$$

coming from comparing atomic clocks at present time ($z = 0$). The overdot denotes differentiation with respect to cosmic time.

3 $f(R)$ Gravity

The action of $f(R)$ gravity in Jordan frame in the presence of matter is given by

$$S = \frac{1}{2} \int d^4x \sqrt{-g} f(R) + \int d^4x \sqrt{-g} L_m(g_{\mu\nu}, \psi) \quad (10)$$

where $f(R)$ is an arbitrary function of the scalar curvature. The Lagrangian density L_m corresponds to matter fields which are collectively denoted by ψ . We apply the following conformal transformation

$$\tilde{g}_{\mu\nu} = \Omega g_{\mu\nu}, \quad \Omega = f'(R) \quad (11)$$

This together with a redefinition of the conformal factor in terms of a scalar field

$$\phi = \frac{1}{2\beta} \ln \Omega \quad (12)$$

[§]For a more complete list of these constraints see, e.g., [12] and references therein.

with $\beta = \sqrt{\frac{1}{6}}$, yields [15]

$$S = \frac{1}{2} \int d^4x \sqrt{-\tilde{g}} \{ \tilde{R} - \tilde{g}^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - V(\phi) \} + \int d^4x \sqrt{-\tilde{g}} e^{-4\beta\phi} L_m(\tilde{g}_{\mu\nu}, \psi) \quad (13)$$

In the action (13), ϕ is minimally coupled to $\tilde{g}_{\mu\nu}$ and appear as a massive self-interacting scalar field with a potential

$$V(\phi) = \frac{1}{2} e^{-2\beta\phi} r[\Omega(\phi)] - \frac{1}{2} e^{-4\beta\phi} f(r[\Omega(\phi)]) \quad (14)$$

where the function $r(\Omega)$ is the solution of the equation $f'[r(\Omega)] - \Omega = 0$ [15]. Thus the variables $(\tilde{g}_{\mu\nu}, \phi)$ provide the Einstein frame variables for $f(R)$ theories. This Einstein frame representation is equivalent to a quintessence model with a specific potential function which, due to (14), is closely related to the function $f(R)$.

It is important to note that in the action (13) the scalar field interacts with matter sector via the function $e^{-4\beta\phi}$. If we take L_m to be the Lagrangian density of electromagnetic field, then the action (13) will become similar to (1) with a gauge function $A(\phi) = e^{-4\beta\phi}$. Thus one can generally state that any $f(R)$ theory in the Einstein frame is equivalent with a quintessence model in which the scalar field interacts with matter sector via an exponential gauge function. However, it should be pointed out that in the model (1) evolution of the scalar field depends on the choice of two arbitrary functions $A(\phi)$ and $V(\phi)$ while in the model (13) it depends only on $V(\phi)$ which in turn is characterized by the function $f(R)$. It implies that depending on cosmological evolution of ϕ , or equivalently on functional form of $f(R)$, the fine structure constant could have many possible histories during evolution of the universe. It is therefore worth studying what possibilities are allowed by the function $f(R)$ within the available observational constraints.

To proceed further, we note that (7) is equivalent to $\alpha = \alpha_0 e^{4\beta\phi}$. We would like to use (11) and (12) to write this relation in terms of scalar curvature rather than the scalar field, namely that $\alpha = \alpha_0 f'^2(R)$ where the prime denotes differentiation with respect to the argument. Thus the rate of change of α is given by

$$\frac{\dot{\alpha}}{\alpha} = 2\dot{R} \frac{f''(R)}{f'(R)} \quad (15)$$

In a spatially flat FRW spacetime $R = 6(\dot{H} + 2H^2)$ and therefore $\dot{R} = 6(\ddot{H} + 4\dot{H}H)$. In terms of deceleration parameter $q \equiv -a\ddot{a}/\dot{a}^2 = -(1 + \dot{H}/H)$ and jerk $j \equiv a^2\ddot{\dot{a}}/\dot{a}^3$ we will have

$$\frac{\dot{\alpha}}{\alpha} = 12(j - q - 2)H^3 \frac{f''(R)}{f'(R)} \quad (16)$$

It is also possible to obtain the relative change of α ,

$$\frac{\Delta\alpha}{\alpha} \equiv \frac{\alpha(z) - \alpha_0}{\alpha_0} \quad (17)$$

which is equivalent to

$$\frac{\Delta\alpha}{\alpha} = f'^2(R) |_z - 1 \quad (18)$$

In this relation $\alpha(z)$ and $f'^2(R) |_z$ indicate fine structure constant and $f'^2(R)$ at redshift z . We can now use (16) and (18) to study the impact of the constraints (8) and (9) on a particular $f(R)$ model.

Let us first consider the model [16]

$$f(R) = R + \lambda R_c \left(\frac{R}{R_c}\right)^n \quad (19)$$

where λ and n are positive parameters and R_c is of the order of the presently observed effective cosmological constant. Here and in the following we will take $R_c = \varepsilon H_0^2$ with ε being a constant of order of unity and H_0 is the Hubble constant. We substitute this $f(R)$ function into (16) and (18) and obtain

$$\frac{\dot{\alpha}}{\alpha} = \frac{12}{\varepsilon} (j - q - 2) H \frac{n(n-1)\lambda x^{n-2}}{1 - n\lambda x^{n-1}} \quad (20)$$

$$\frac{\Delta\alpha}{\alpha} = n\lambda x^{n-1} (n\lambda x^{n-1} + 2) \quad (21)$$

where $x \equiv R/R_c = \frac{6}{\varepsilon}(1-q)h^2$ with $h = H(z)/H_0$. Using the observational fact that $H_0 = 72 \pm 8 \text{ km}^{-1} \text{ s Mpc}^{-1} = 7.4 \times 10^{-10} \text{ yr}^{-1}$ [17], then the experimental bounds (8) and (9) imply

$$\frac{\dot{\alpha}}{\alpha} |_{z=0} = 12(j_0 - q_0 - 2) \frac{n(n-1)\lambda x_0^{n-2}}{1 + n\lambda x_0^{n-1}} < 10^{-5} \quad (22)$$

$$\frac{\Delta\alpha}{\alpha} |_{z=0.45} = n\lambda x^{n-1} (n\lambda x^{n-1} + 2) < 10^{-7} \quad (23)$$

where we have set $\varepsilon = 1$ and the subscript 0 represents the present value of a quantity. It is now evident that the relations (22) and (23) act as algebraic constraints on the parameters λ and n for given values of the quantities j_0 , q_0 , $q(z = 0.45)$ and $H(z = 0.45)$.

To estimate these quantities, we use the gold data set for j_0 and q_0 which gives $j_0 = 2.75_{-1.10}^{+1.22}$ and $q_0 = -0.86 \pm 0.21$ [18]. On the other hand, the Hubble parameter H and the deceleration parameter q are related by

$$H(z) = H_0 \exp \left[\int_0^z (1 + q(u)) d \ln(1 + u) \right] \quad (24)$$

If a function $q(z)$ is given, then we can find the Hubble parameter in the redshift z . Here we use a two-parametric reconstruction function characterizing $q(z)$ [19][20],

$$q(z) = \frac{1}{2} + \frac{q_1 z + q_2}{(1 + z)^2} \quad (25)$$

where fitting the model to the gold data set gives $q_1 = 1.47_{-1.82}^{+1.89}$ and $q_2 = -1.46 \pm 0.43$ [20]. Using this in (24) yields [20]

$$H(z) = H_0 (1 + z)^{3/2} \exp \left[\frac{q_2}{2} + \frac{q_1 z^2 - q_2}{2(z + 1)^2} \right] \quad (26)$$

At $z = 0.45$, (25) and (26) give $q = 0.12$ and $H = 1.28 H_0$. As an illustration, the changes of $\frac{\dot{\alpha}}{\alpha}$ and $\frac{\Delta\alpha}{\alpha}$ in terms of the parameters λ and n are plotted in Fig.1. The figure shows that

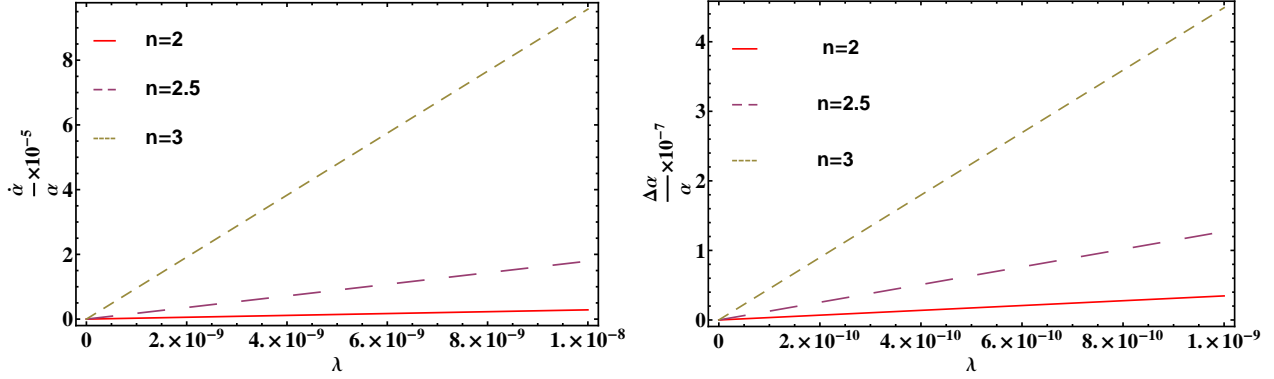


Figure 1: The changes of $\frac{\dot{\alpha}}{\alpha}$ and $\frac{\Delta\alpha}{\alpha}$ in terms of n and λ for the model (19). The vertical axes are scaled according to the observational bound. The figures show that for larger values of n the slope of the lines increase. This implies that for moderate values of n , λ should be extremely small.

the bounds (22) and (23) are satisfied for small values of the parameters. For instance, for λ to be of the order of unity one should have $n < 10^{-4}$. A moderate value of n requires an extremely small value for λ which makes the model (19) to be hardly distinguishable from Einstein gravity.

Now we consider the models presented by Starobinsky [21]

$$f(R) = R - \lambda R_c \{1 - [1 + (\frac{R}{R_c})^2]^{-n}\} \quad (27)$$

and Hu-Sawicki [22]

$$f(R) = R - \lambda R_c \left\{ \frac{(\frac{R}{R_c})^n}{1 + (\frac{R}{R_c})^n} \right\} \quad (28)$$

which can be written in a unified form [23]

$$f(R) = R - \lambda R_c \{1 - [1 + (\frac{R}{R_c})^n]^{-\frac{1}{\beta}}\} \quad (29)$$

This parametrization corresponds to (27) and (28) with $n = 2$, and $\beta = 1$, respectively. To apply the experimental bounds, we use (16) and (18) and obtain

$$\frac{\dot{\alpha}}{\alpha} \Big|_{z=0} = 12(j_0 - q_0 - 2) \frac{\frac{\lambda n}{\beta} x_0^{n-2} (1 + x_0^n)^{-\frac{1}{\beta}-1} [n(\frac{1}{\beta} + 1)x_0^n (1 + x_0^n)^{-1} - (n-1)]}{1 - \frac{\lambda n}{\beta} x_0^{n-1} (1 + x_0^n)^{-\frac{1}{\beta}-1}} < 10^{-5} \quad (30)$$

$$\frac{\Delta\alpha}{\alpha} \Big|_{z=0.45} = \frac{\lambda n}{\beta} x^{n-1} (x^n + 1)^{-\frac{1}{\beta}-1} \left\{ \frac{\lambda n}{\beta} x^{n-1} (x^n + 1)^{-\frac{1}{\beta}-1} - 2 \right\} < 10^{-7} \quad (31)$$

We have used the same data set for the quantities j_0 , q_0 , $q(z = 0.45)$ and $H(z = 0.45)$ and plotted (30) and (31) in Fig. 2. The bounds satisfy in the region of the parameter space which

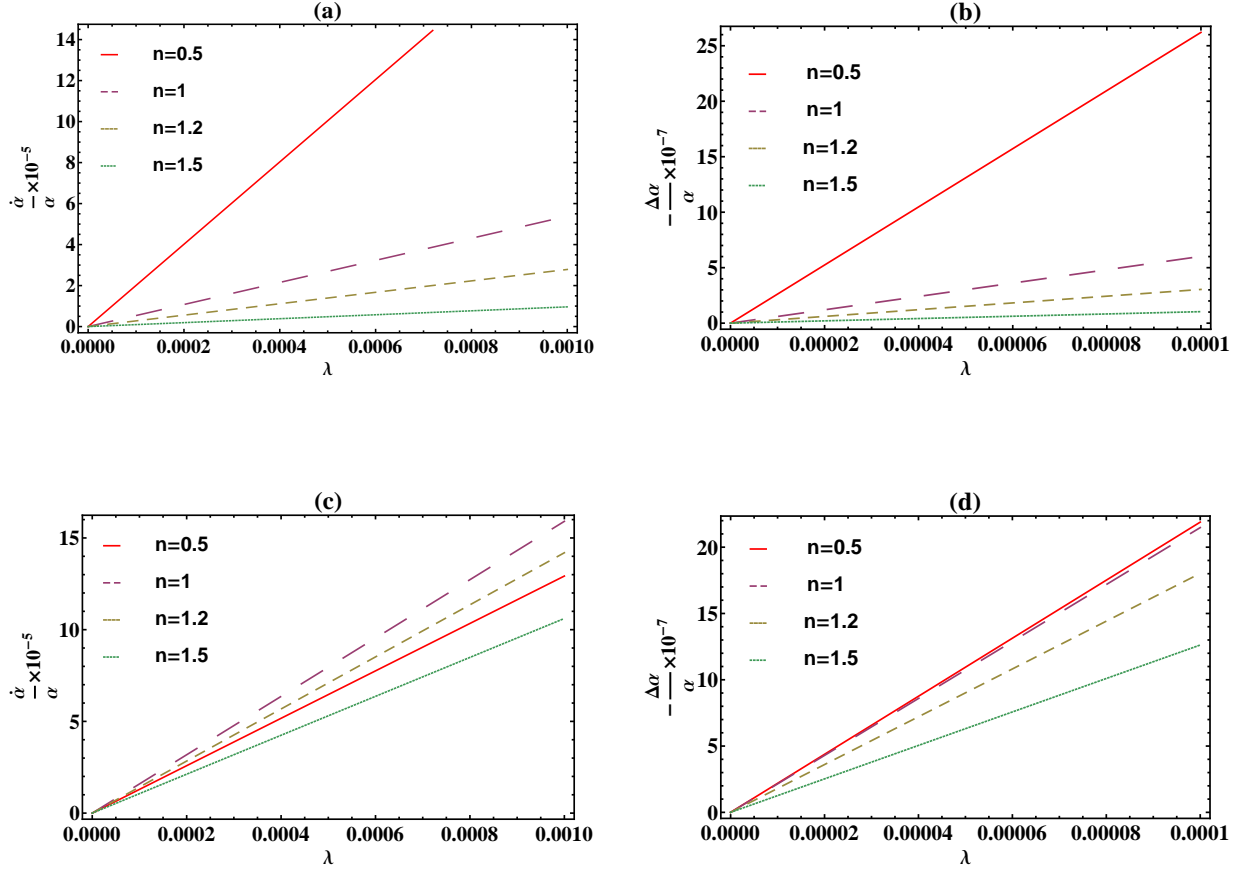


Figure 2: The changes of $\frac{\dot{\alpha}}{\alpha}$ and $\frac{\Delta\alpha}{\alpha}$ for the Starobinsky (panels a, b) and Hu-Sawicki (panels c, d) models. The vertical axes are scaled according to the observational bound. In contrary to the model (19), in these models the larger values of n correspond to decrease of the slope of the lines.

corresponds to larger values of n and smaller values of λ . For a given value of n , the acceptable range of λ is $[0, \lambda_1]$ with λ_1 being the maximum value of λ which satisfies the experimental bounds. If we take larger values of n the bounds (30) and (31) allow a larger λ_1 which means that the whole range of acceptable λ is larger. The figures show that for a given n ($n > 1$), λ_1 for the model (28) is smaller than that for the model (27). For n near unity, the acceptable values of λ are very small for the both models.

4 Conclusion

A possible modification of gravity is replacing the curvature scalar in the Einstein-Hilbert action by a general $f(R)$ function. It is well-known that these $f(R)$ gravity models are conformally equivalent with a class of quintessence models in which gravity is minimally coupled with

a scalar field with some appropriate potential function. In this Einstein frame representation of $f(R)$ models, conformal transformations induce a coupling of the scalar field with ordinary matter which may potentially lead to variation of constants of nature.

Although both quintessence models and $f(R)$ theories in the Einstein frame can lead to changes of fundamental constants, there is a basic difference between these two alternatives. In the quintessence models, the coupling of the scalar field with matter system is characterized by an arbitrary gauge function $A(\phi)$. Cosmological evolution of ϕ is then determined by $A(\phi)$ and some potential function of the scalar field $V(\phi)$. On the other hand, we have shown that in $f(R)$ theories the possible variation of constants only depends on the functional form of $f(R)$. This is due to the fact that the gauge function is fixed by conformal transformations and is given by an exponential function and the potential of the scalar field is determined by $f(R)$ itself.

Based on these arguments we have introduced a criterion that can be used to probe the cosmological viability of $f(R)$ theories. In this criterion, we have considered a general $f(R)$ theory in the Einstein frame and restrict ourselves to temporal variations of fine structure constant. We have shown that for any $f(R)$ gravity model variations of fine structure constant can be written in terms of some cosmological parameters (such as q , j and H) in an appropriate redshift. In particular, the relations (16) and (18) allows us to apply the experimental bounds on the rate and the relative changes of α for any $f(R)$ gravity model. We have then used the gold data set to estimate q , j and H at redshifts $z = 0$ and $z = 0.45$ for applying to some particular $f(R)$ functions.

Specifically, we have applied the above procedure to models of Starobinsky and Hu-Sawicki. We have shown that these two-parameter models satisfy the experimental bounds in the regions of parameter space which correspond to larger values of n and smaller values of λ . This result is more restrictive than the constraints coming from local gravity experiments which set a lower bound only on n [24]. In our analysis both the parameters n and λ should satisfy the algebraic constraints (30) and (31) so that a particular value for one parameter affect the whole of range of validity of the other. Moreover, we have shown that for a given value of n , there is an upper bound on the allowed range of λ (λ_1). For $n = 1$, λ_1 will be of the order of 10^{-4} in both models. It is important to note that the local gravity analysis gives $n > 0.5$ as the allowed range of n [24]. Although the outcome of our analysis is consistent with this result, it however emphasizes that this parameter should take values sufficiently larger than unity in order that the models keep an appropriate departure from Einstein gravity.

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